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UNSTEADY-STATE CONJUGATED HEAT TRANSFER BETWEEN A SEMI-INFINITE SURFACE AND INCOMING FLOW OF A COMPRESSIBLE FLUID—II. DETERMINATION OF A TEMPERATURE FIELD AND ANALYSIS OF RESULTS

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Abstract—The analytical solution of the conjugated unsteady heat transfer problem in a semi-infinite plate with the sources in the form of the generalized power series is obtained in the space of the generalized functions (in the Sobolev–Schwartz sense).

The dimensionless parameter $B = [(1/Re).(a_s/ca_\infty)]$ having physical meaning is obtained which allows consideration of the problem as the unsteady or quasi-stationary one depending on the "body–liquid" pair.

8. SOLUTION OF RECURRENT RELATIONS

Due to the linearity, further consider the case $R_i = R_n\delta_{ni}$. We shall search for $\bar{y}_k^{(1)}$, $\varphi_e^{(1)}$, $y_k^{(1)}$, $\bar{\varphi}_e^{(1)}$, $\bar{y}_k^{(2)}$, $\bar{\varphi}_e^{(2)}$, $y_k^{(2)}$, $\varphi_e^{(2)}$ in the form respectively

$$\bar{y}_k^{(1)} = \sum_{s=0}^{k-n-1} \alpha_{k,s,n} \frac{d^s R_n}{d\tau^s}, \tag{89}$$

$$y_k^{(1)} = \sum_{s=0}^{k-n-1} l_{k,s,n} \frac{d^s R_n}{d\tau^s}, \tag{90}$$

$$\bar{\varphi}_k^{(1)} = \sum_{s=0}^{k-n-3} \beta_{k,s,n} \frac{d^s R_n}{d\tau^s}, \tag{91}$$

$$\varphi_k^{(1)} = \sum_{s=0}^{k-n-3} \gamma_{k,s,n} \frac{d^s R_n}{d\tau^s}, \tag{92}$$

$$\bar{y}_k^{(2)} = \sum_{s=0}^{k+1} \alpha_{k,s} \frac{d^s \bar{y}_{-1}^{(2)}}{d\tau^s}, \tag{93}$$

$$\bar{\varphi}_k^{(2)} = \sum_{s=0}^{k-1} \beta_{k,s} \frac{d^s \bar{y}_{-1}^{(2)}}{d\tau^s}, \tag{94}$$

$$y_k^{(2)} = \sum_{s=0}^{k+1} l_{k,s} \frac{d^s \bar{y}_{-1}^{(2)}}{d\tau^s}, \tag{95}$$

$$\varphi_k^{(2)} = \sum_{s=0}^{k-1} \gamma_{k,s} \frac{d^s \bar{y}_{-1}^{(2)}}{d\tau^s}. \tag{96}$$

The recurrent relations for successive calculation of the coefficients entering into formulae (89)–(96) are obtained. Having substituted (89)–(92) into (72)–(75) upon some transformations we arrive at:

$$\beta_{k,s,n} = \sum_{j=0}^{s} (-B)^j \gamma_{k-j,s-j,n} \varkappa_{jk}^{(1)}(0), \tag{97}$$

$$\gamma_{2m,s,n} = -\frac{M}{\Gamma(\rho + 2m + \frac{3}{2})} \times \sum_{i=i_1}^{m} \Gamma(\rho + 2i - \tfrac{1}{2}) \alpha_{2i-2,s+i-m,n} \tag{98}$$

where†

$$i_1 = \begin{cases} m - s & \text{at } s \leqslant m - \tilde{n}_1; \\ s + n + 3 - m & \text{at } m - \tilde{n}_1 + 1 \leqslant s; \end{cases}$$

$$\tilde{n}_1 = \frac{n+3}{2} + \left\{\frac{n+3}{2}\right\}$$

$$\gamma_{2m-1,s,n} = -\frac{M}{\Gamma(\rho + 2m + \frac{1}{2})} \sum_{i=i_2}^{m} \Gamma(\rho + 2i - \tfrac{3}{2})\,\alpha_{2i-3,\,s+i-m,\,n}, \qquad (99)$$

where

$$i_2 = \begin{cases} m - s & \text{at } s \leqslant m - \tilde{n}_2; \\ s + n + 4 - m & \text{at } m - \tilde{n}_2 + 1 \leqslant s, \end{cases}$$

$$\tilde{n}_2 = \frac{n+4}{2} + \left\{\frac{n+4}{2}\right\},$$

$$\alpha_{k,s,n} = \frac{1}{\psi_{0,k}^{(1)}(0)}\left(l_{k,s,n} - \sum_{j=1}^{s} (-B)^j \times \alpha_{k-j,\,s-j,\,n}\psi_{jk}^{(1)}(0)\right). \qquad (100)$$

Using (69) we find:

$$l_{k,s,n} = \frac{l_{k-2,s-1,n} - M\beta_{k-1,s,n} - \delta_{0,s}\cdot\delta_{k-1,n}}{(\rho + k)(\rho + k + 1)}. \qquad (101)$$

Similarly, having substituted (93)–(96) into (80), (82)–(85), we have the following relations:

$$\beta_{k,s} = \sum_{j=0}^{s} (-B)^j \gamma_{k-j,\,s-j} \varkappa_{j,k}^{\prime(2)}(0), \qquad (102)$$

$$\gamma_{2m,s} = -\frac{M}{\Gamma(2m + \frac{3}{2})} \sum_{i=\tilde{i}_1}^{m} \Gamma(2i - \tfrac{1}{2})\,\alpha_{2i-2,\,s+i-m}, \qquad (103)$$

where

$$\tilde{i}_1 = \begin{cases} m - s & \text{at } s \leqslant m - 1, \\ s + 1 - m & \text{at } m \leqslant s \end{cases}$$

$$\gamma_{2m-1,s} = -\frac{M}{\Gamma(2m + \frac{1}{2})} \sum_{i=\tilde{i}_2} \Gamma(2i - \tfrac{3}{2}) \times \alpha_{2i-3,\,s+i-m}, \qquad (104)$$

† $\{a\}$ denotes the fractional part of a.

where

$$\tilde{i}_2 = \begin{cases} m - s & \text{at } s \leqslant m - 1, \\ s + 2 - m & \text{at } m \leqslant s, \end{cases}$$

$$\alpha_{k,s} = \frac{1}{\psi_{0,k}^{(2)}(0)}\left(l_{k,s} - \sum_{j=1}^{s} (-B)^j \alpha_{k-j,\,s-j} \times \psi_{jk}^{(2)}(0)\right), \qquad (105)$$

$$l_{k,s} = \frac{1}{k(k+1)}(l_{k-2,s-1} - M\beta_{k-1,s}), \; k = 1, 2, \ldots \qquad (106)$$

From (93) at $k = -1$ it follows: $\alpha_{1,0} = 1$. To fulfil the condition $y_0^{(2)} = 0$ it is necessary: $l_{0,s} \equiv 0$. (Note that from (105) it follows: $l_{-1,0} = \psi_{0,-1}^{(2)}(0)$.)

Using these conditions together with recurrent relations (97)–(106), it is possible to calculate successively all the coefficients, entering into formulae (89)–(96). For the quasistationary case relations (97), (100), (102) and (105) become essentially simpler and assume the following form:

$$\beta_{k,s,n} = \gamma_{k,s,n}\varkappa_{0,k}^{\prime(1)}(0), \qquad (97')$$

$$\alpha_{k,s,n} = l_{k,s,n}/\psi_{0,k}^{(1)}(0), \qquad (100')$$

$$\beta_{k,s} = \gamma_{k,s}\varkappa_{0,k}^{\prime(2)}(0), \qquad (102')$$

$$\alpha_{k,s} = l_{k,s}/\psi_{0,k}^{(2)}(0). \qquad (105')$$

9. SOLUTION OF INTEGRAL RELATION

To study integral relation (88), apply the Laplace transform to it [1]. If it is taken into account, that

$$\pi^{-\frac{1}{2}}\tau^{-\frac{1}{2}} \exp\left\{-\frac{z^2}{4\tau}\right\} \doteqdot p^{-\frac{1}{2}} \exp\{-p^{\frac{1}{2}} z\},$$

$$\hat{f} \overset{\text{def}}{=} \int_0^\infty f \cdot e^{-p\tau}\, d\tau,$$

then from (88) we have:

$$l_{-1,0}\hat{\bar{y}}^{(2)}_{-1}(p) = \int_0^\infty \left[\frac{M}{\sqrt{z}}\frac{\partial\hat{\Theta}(z,0,p)}{\partial\eta} + \hat{R}(z,p)\right] \times \frac{e^{-\sqrt{pz}}}{\sqrt{p}}\,dz. \quad (107)$$

After taking into account (79) and that

$$\hat{R}(z,p) = \sum_{n=0}^{\infty}\hat{R}_n(p)\,z^{\rho+n},$$

$$\int_0^\infty z^a p^{-\frac{1}{2}} e^{-p^{\frac{1}{2}}z}\,dz = p^{-(a+2)/2}\Gamma(a+1), \quad a > -1$$

expression (107) assumes the form:

$$l_{-1,0}\hat{\bar{y}}^{(2)}_{-1}(p) = M\left\{\sum_{k=1}^{\infty}\hat{\bar{y}}^{(1)}_k(p)\frac{\Gamma(\rho+k+\frac{3}{2})}{p^{(\rho+k+\frac{3}{2})/2}} + \sum_{l=3}^{\infty}\hat{\bar{\varphi}}^{(1)}_l(p)\frac{\Gamma(\rho+l+1)}{p^{(\rho+l+2)/2}} + \sum_{k=-1}^{\infty}\hat{\bar{y}}^{(2)}_k(p)\frac{\Gamma(k+\frac{3}{2})}{p^{(k+\frac{3}{2})/2}} + \sum_{l=1}^{\infty}\hat{\bar{\varphi}}^{(2)}_l(p)\frac{\Gamma(l+1)}{p^{(l+2)/2}}\right\} + \sum_{n=0}^{\infty}\hat{R}_n(p)\frac{\Gamma(\rho+n+1)}{p^{(\rho+n+2)/2}}. \quad (108)$$

Pass to the Laplace transform in (89), (91), (93) and (94)

$$\hat{\bar{y}}^{(1)}_k(p) = \hat{R}_n(p)\sum_{s=0}^{k-n-1}\alpha_{k,s,n}p^s, \quad (109)$$

$$\hat{\bar{\varphi}}^{(1)}_l(p) = \hat{R}_n(p)\sum_{s=0}^{l-n-3}\beta_{l,s,n}p^s, \quad (110)$$

$$\hat{\bar{y}}^{(2)}_k(p) = \hat{\bar{y}}^{(2)}_{-1}(p)\sum_{s=0}^{k+1}\alpha_{k,s}p^s, \quad (111)$$

$$\hat{\bar{\varphi}}^{(2)}_l(p) = \hat{\bar{y}}^{(2)}_{-1}(p)\sum_{s=0}^{l-1}\beta_{l,s}p^s. \quad (112)$$

Having substituted (109)–(112) into (108) we obtain:

$$\hat{\bar{y}}^{(2)}_{-1}(p) = \hat{\Omega}_n(p).\hat{R}_n(p). \quad (113)$$

Passing in (113) to the inverted transform and taking into account that $R_n(\tau)\in K$. We have

$$\bar{y}^{(2)}_{-1}(\tau) = \Omega_n(\tau) * R_n(\tau) = (\Omega_n(\tau'), R_n(\tau-\tau'))\in C^\infty \quad (114)$$

(see [I]).

Since $\Omega_n(\tau)$ is the regular generalized function, then after substitution of (114) into (93)–(96) we have:

$$\bar{y}^{(2)}_k = \sum_{s=0}^{k+1}\alpha_{k,s}\int_0^\tau \Omega_n(\tau-\tau')\frac{d^sR_n(\tau')}{d\tau'^s}\,d\tau', \quad (115)$$

$$\bar{\varphi}^{(2)}_k = \sum_{s=0}^{k-1}\beta_{k,s}\int_0^\tau \Omega_n(\tau-\tau')\frac{d^sR_n(\tau')}{d\tau'^s}\,d\tau' \quad (116)$$

$$y^{(2)}_k = \sum_{s=0}^{k+1}l_{k,s}\int_0^\tau \Omega_n(\tau-\tau')\frac{d^sR_n(\tau')}{d\tau'^s}\,d\tau', \quad (117)$$

$$\varphi^{(2)}_k = \sum_{s=0}^{k-1}\gamma_{k,s}\int_0^\tau \Omega_n(\tau-\tau')\frac{d^sR_n(\tau')}{d\tau'^s}\,d\tau'. \quad (118)$$

Thus, $y^{(2)}_k$, $\varphi^{(2)}_k$, $\bar{y}^{(2)}_k$, $\bar{\varphi}^{(2)}_k$ and, consequently, $\Theta^{(2)}$ are found (see (43)–(54')).

10. APPROXIMATE CALCULATION OF $\Omega_n(\tau)$

Approximation will be carried out by the number of the terms in the series $\Theta^{(1)}$ and $\Theta^{(2)}$. Assuming that $y^{(2)}_{-1}(\tau)$ is unknown, choose for $\Theta^{(2)}$ the N successive terms, starting from the first non-vanishing term $(k = 1, 2, \ldots, N; l = 1, 2, \ldots, N)$. With regard for (111) and (112) we have:

$$\sum_{k=-1}^{N}\hat{\bar{y}}^{(2)}_k(p)\frac{\Gamma(k+\frac{3}{2})}{p^{(k+\frac{3}{2})/2}} = \frac{\hat{\bar{y}}^{(2)}_{-1}(p)}{p^{\frac{1}{4}}} \times \sum_{m=-N}^{N+2}\sigma_m p^{m/2}, \quad (119)$$

where†

$$\sigma_m = \sum_{s=s_1}^{[(N+m)/2]} \alpha_{2s-m,s}\, \Gamma(2s - m + \tfrac{3}{2});$$

$$s_1 = \begin{cases} 0 \text{ at } m < 1, \\ m - 1 \text{ at } m \geqslant 1. \end{cases}$$

$$\sum_{l=1}^{N} \hat{\bar{\varphi}}_l^{(2)}(p) \frac{\Gamma(l+1)}{p^{(l+2)/2}} = \frac{\hat{\bar{y}}_{-1}^{(2)}(p)}{p} \sum_{m=-N}^{N-2} \delta_m\, p^{m/2}, \tag{120}$$

where

$$\delta_m = \sum_{s=s_2}^{[(N+m)/2} \beta_{2s-m,s}\, \Gamma(2s - m + 1);$$

$$s_2 = \begin{cases} m + 1 \text{ at } m \geqslant -1, \\ 0 \quad\quad \text{at } m < -1. \end{cases}$$

For $\Theta^{(1)}$ choose $k = n+1, \ldots, n+N$; $l = n+3, \ldots, n+N+2$. Using (109) and (110), we arrive at

$$\sum_{k=n+1}^{n+N} \hat{\bar{y}}_k^{(1)}(p) \frac{\Gamma(\rho + k + \frac{3}{2})}{p^{(\rho+k+\frac{5}{2})/2}} = \frac{\hat{R}_n(p)}{p^{(\rho+n+\frac{5}{2})/2}} \sum_{m=-N}^{N-2} \sigma_{m,n}\, p^{m/2}, \tag{121}$$

where

$$\sigma_{m,n} = \sum_{s=s_2}^{[(N+m)/2]} \alpha_{2s-m+n,s,n}\, \Gamma(2s - m + n + \rho + \tfrac{3}{2}).$$

$$\sum_{l=n+3}^{n+N+2} \bar{\varphi}_l^{(1)}(p) \frac{\Gamma(\rho + l + 1)}{p^{(\rho+l+2)/2}} = \frac{\hat{R}_n(p)}{p^{(\rho+n+4)/2}} \sum_{m=-N}^{N-2} \delta_{m,n}\, p^{m/2}, \tag{122}$$

where

$$\delta_{m,n} = \sum_{s=s_2}^{[(N+m)/2]} \beta_{2s-m+n+2,s,n}\, \Gamma(2s - m + n + 3 + \rho).$$

Replacing in (108) the infinite sums by the finite sums (119)–(122) we obtain approximately

$$\hat{\Omega}_n(p) = \frac{T_n(p^{\frac{1}{2}})}{p^{(\rho+n+\frac{3}{2})/2}\, \Pi(p^{\frac{1}{2}})}, \tag{123}$$

† $[a]$ denotes the integral part of the number a.

where

$$T_n(u) = \sum_{m=-N}^{N-2} M\sigma_{m,n}\, u^{2(m+N)+3} + \sum_{m=-N}^{N-2} M\delta_{m,n}\, u^{2(m+N)} + \Gamma(\rho+n+1)u^{2N+4}, \tag{124}$$

$$\Pi(u) = l_{-1,0}\, u^{2N+5} - \sum_{m=-N}^{N+2} M\delta_m u^{2(m+N)} - \sum_{m=-N}^{N-2} M\delta_m u^{2(m+N)+1}. \tag{125}$$

Pass to the inverted transform in (123). In calculations use twice the formula from ([2], Chapter V):

$$p^{\nu-1}\, \hat{g}(p^{\frac{1}{2}}) \doteqdot \frac{1}{\tau^\nu 2^{\nu-\frac{1}{2}} \pi^{\frac{1}{2}}} \int_0^\infty e^{-(x^2/8\tau)} D_{2\nu-1}\left(\frac{x}{\sqrt{(2\tau)}}\right) g(x)\, dx$$

($D_\nu(x)$ is the function of the parabolic cylinder). Let

$$\Pi(u) = l_{1,0}(u - r_1)^{m_1}, \ldots, (u - r_\sigma)^{m_\sigma},$$

$$\frac{T_n(p)}{\Pi(p)} \doteqdot \sum_{k=1}^{\sigma} \sum_{l=1}^{m_k} \frac{c_{n,k,l}(r_k)}{(m_k - l)!\,(l-1)!} \tau^{m_k - l}\, e^{r_k \tau} \overset{\text{def}}{=} g_n(\tau), \tag{126}$$

where

$$c_{n,k,l}(p) = \frac{d^{l-1}}{dp^{l-1}}\left[\frac{T_n(p)}{\Pi_k(p)}\right], \quad \Pi_k(p) = \frac{\Pi(p)}{(p - r_k)^{m_k}},$$

$$\frac{T_n(p^{\frac{1}{2}})}{\Pi(p^{\frac{1}{2}})} \doteqdot \frac{1}{\sqrt{2\pi\tau}} \int_0^\infty e^{-(x^2/8\tau)} D_1\left(\frac{x}{\sqrt{(2\tau)}}\right) g_n(x)\, dx \overset{\text{def}}{=} G_n(\tau) \tag{127}$$

$$\Omega_n(\tau) = \pi^{-\frac{1}{2}}\, 2^{(\rho+n+\frac{1}{2})/2}\, \tau^{(\rho+n+\frac{1}{2})/2} \int_0^\infty e^{-(x^2/8\tau)} D_{-(\rho+n+\frac{1}{2})}\left(\frac{x}{\sqrt{(2\tau)}}\right) G_n(x)\, dx. \tag{128}$$

In Appendix III it is shown that $\Omega_n(\tau)$ is represented by the power series converging at any τ

$$\Omega_n(\tau) = \tau^{(\rho+n)/2} \sum_{s=0}^{\infty} \lambda_{n,s} \tau^{s/4},$$

where $\quad \lambda_{n,s} = \sum_{k=1}^{\sigma} \lambda_{n,k,s}.$

11. ANALYSIS OF RESULTS

So, to the N-th approximation the plate temperature and the heat flux through it has the form:

$$T_s(z,\tau) = T_\infty \Bigg[\sum_{k=n+1}^{n+N} y_k^{(1)}(\tau)\, z^{\rho+k+1} + \sum_{l=n+3}^{n+N+2} \varphi_l^{(1)}(\tau)\, z^{\rho+l+\frac{1}{2}} + \sum_{k=-1}^{N} y_k^{(2)}(\tau)\, z^{k+1} + \sum_{l=1}^{N} \varphi_l^{(2)} z^{l+\frac{1}{2}} + N(0) \Bigg], \tag{129}$$

$$q_s(z,\tau) = \lambda_\infty \frac{T_\infty}{L} \left(\frac{CRe}{2}\right)^{\frac{1}{2}} \Bigg[\sum_{k=n+1}^{n+N} \bar{y}_k^{(1)}(\tau) z^{\rho+k+\frac{1}{2}} + \sum_{l=n+3}^{n+N+2} \bar{\varphi}_l^{(1)}(\tau) z^{\rho+l} + \sum_{k=-1}^{N} \bar{y}_k^{(2)}(\tau) z^{k+\frac{1}{2}} + \sum_{l=1}^{N} \bar{\varphi}_l^{(2)}(\tau) z^l \Bigg]. \tag{130}$$

It is easy to calculate the Nusselt number

$$Nu(z,\tau) = -\frac{L}{(T_s(z,\tau) - T_\infty)\lambda_\infty} \frac{q_s(z,\tau)}{}.$$

For small z from (129) and (130) with the use of formula (II. 24) we have

$$\frac{q_s(z,\tau)\sqrt{z}}{T_s(0,\tau) - T_\infty N(0)} \approx -\frac{\lambda_\infty}{L}\left(\frac{CRe}{2}\right)^{\frac{1}{2}} \frac{1}{I_{-1}^{(2)}\{F\}}$$

Consider in detail the behaviour of $T_s(z,\tau)$ and $q_s(z,\tau)$ at $\tau > T$ where T is the moment of the end of operation of a quick-response source. By a quick-response source is understood such a source, whose coefficients $R_k(\tau)$ satisfy the following condition: at any $L(\tau) \subset (k)$ and $L_1(\tau)$

$$\subset K' \left| \left(\frac{R_k(\tau)}{\tilde{R}_k} * L_1, L \right) - (L_1(\tau), L(\tau)) \right| < \varepsilon$$

where ε is some small number. $\tilde{R}_k = \int_0^T R_k(\tau)\, d\tau$ is the impulse of the source coefficient. Then, from (115)–(118), (129) and (130) we have:

$$\bar{y}_k^{(2)} \approx \tilde{R}_n \sum_{s=0}^{k+1} \alpha_{k,s} \frac{d^s\Omega_n}{d\tau^s}, \tag{131}$$

$$\bar{\varphi}_k^{(2)} \approx \tilde{R}_n \sum_{s=0}^{k-1} \beta_{k,s} \frac{d^s\Omega_n}{d\tau^s}, \tag{132}$$

$$y_k^{(2)} \approx \tilde{R}_n \sum_{s=0}^{k+1} l_{k,s} \frac{d^s\Omega_n}{d\tau^s}, \tag{133}$$

$$\varphi_k^{(2)} \approx \tilde{R}_n \sum_{s=0}^{k-1} \gamma_{k,s} \frac{d^s\Omega_n}{d\tau^s}, \tag{134}$$

$$T_s(z,\tau) = T_\infty \Bigg[\sum_{k=-1}^{N} y_k^{(2)}(\tau) z^{k+1} + \sum_{l=1}^{N} \varphi_l^{(2)}(\tau) z^{l+\frac{1}{2}} + N(0) \Bigg], \tag{135}$$

$$q_s(z,\tau) = \lambda_\infty \frac{T_\infty}{L}\left(\frac{CRe}{2}\right)^{\frac{1}{2}} \Bigg[\sum_{k=-1}^{N} \bar{y}_k^{(2)}(\tau) z^{k+\frac{1}{2}} + \sum_{l=1}^{N} \bar{\varphi}_l^{(2)}(\tau) z^l \Bigg]. \tag{136}$$

From (133), (135) and (II.24) we find $T_s(0,\tau) \approx T_\infty(N(0) - \tilde{R}_n I_{-1}^{(2)}\{F\}\,\Omega_n(\tau))$ i.e. $\Omega_n(\tau)$ has simple physical meaning. It determines the dependence of a temperature at the leading edge upon time after switching off a quick-response source.

At last, note that using the asymptotic expressions for $\psi_{jk}^{(i)}(\eta)$, $\varkappa_{jk}^{(i)}(\eta)$ at small and large η it is not difficult to obtain simple asymptotic expressions for a temperature field near and far from a plate

$$\Theta^{(i)} = \sum_{n=0}^{\infty} \omega_n^{(i)}(\xi,\tau)\,\eta^n, \qquad \text{at} \qquad \eta \to 0,$$

$$\omega_n^{(i)}(\xi,\tau) = \sum_{m=0}^{n} \Bigg\{ \sum_k \Bigg[\sum_{j=0}^{k+1} (-B)^j \times \frac{d^j \bar{y}_{k-j}^{(i)}}{d\tau^j} \frac{r_{j,k,m}^{(i)} \zeta_{1,k,n-m}^{(i)}}{m!\,(n-m)!} \Bigg] \xi^{2(\rho^{(i)}+k+1)} + \sum_l \Bigg[\sum_{j=0}^{l+1} (-B)^j \frac{d^j \varphi_{l-j}^{(i)}}{d\tau^j} \frac{s_{j,l,m}^{(i)} \zeta_{2,l,n-m}^{(i)}}{m!(n-m)!} \Bigg] \xi^{2(\rho^{(i)}+l+\frac{1}{2})} \Bigg\}, \tag{137}$$

where

$$\zeta^{(i)}_{1,s,m} = \frac{\mathrm{d}^m}{\mathrm{d}\eta^m}\left\{\exp\left[\frac{Pr}{2}(E^{(i)}_s - F)\right]\right\}\Bigg|_{\eta=0},$$

when $s = k$ $E^{(i)}_k \overset{\text{def}}{=} F^{(i)}_k$, and when $s = l$ $E^{(i)}_l \overset{\text{def}}{=} P^{(i)}_l$.

$$\Theta^{(i)} = A^{(i)}(\xi,\eta,\tau)\frac{\exp\left\{-\frac{Pr}{2}(\eta - B)^2\right\}}{Pr\,\eta} \quad \text{at} \quad \eta \to \infty, \tag{138}$$

$$A^{(i)}(\xi,\eta,\tau) = \sum_l A^{(i)}_l\left[\sum_{j=0}^{l+1}\left(-\frac{B\eta^2}{2Pr}\right)^j \times \frac{1}{j!\,I^{(i)}_{l-j}\{P\}}\frac{\mathrm{d}^j\bar{\varphi}^{(i)}_{l-j}}{\mathrm{d}\tau^j}\right]\left(\frac{\xi}{\eta}\right)^{2(\rho^{(i)}+l+\frac{1}{2})} - \sum_k C^{(i)}_k\left[\sum_{j=0}^{k+1}\left(-\frac{B\eta^2}{2Pr}\right)^j\frac{1}{j!}\frac{\mathrm{d}^j\bar{y}^{(i)}_{k-j}}{\mathrm{d}\tau^j}\right]\left(\frac{\xi}{\eta}\right)^{2(\rho^{(i)}+k+1)},$$

$A^{(i)}_l$, $C^{(i)}_k$ are determined when matching series (137) and (138).

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APPENDIX I

The solution of equation (44) where for small η [3]

$$f(\eta) = \frac{\alpha\eta^2}{2!} - \frac{\alpha^2\eta^5}{5!} + \frac{11\alpha^3\eta^8}{11!} - \frac{375\alpha^4\eta^{11}}{11!} + \frac{27897\alpha^5\eta^{14}}{14!} -$$

is sought in the form $\alpha = 0,4696\ldots$,

$$q^{(i)}_m(\eta) = \sum_{n=2}^{+\infty} B^{(i)}_{m,n}\eta^n, \tag{I.1}$$

Upon substitution of (I.1) into (44) we have

$$B^{(i)}_{m,2} = \frac{\lambda^{(i)}_m\alpha}{2},\ B^{(i)}_{m,3} = 0,\ B^{(i)}_{m,4} = 0,\ B^{(i)}_{m,5} = -\frac{\lambda^{(i)}_m\alpha^2}{120} + \frac{Pr\alpha^2(1 - \lambda^{(i)2}_m)}{40},\ B^{(i)}_{m,6} = 0,\ B^{(i)}_{m,7} = 0,$$

$$B^{(i)}_{m,8} = \frac{11\lambda^{(i)}_m\alpha^3}{40320} + \frac{Pr\alpha^3(\lambda^{(i)2}_m - 1)(1 + 3Pr\lambda^{(i)}_m)}{1920},$$

$$\lambda^{(i)}_m \overset{\text{def}}{=} 1 + 2\mu^{(i)}_m. \tag{I.2}$$

Further, we need the expansion

$$Q^{(i)}_m(\eta) = \int_0^\eta q^{(i)}_m(\eta)\,\mathrm{d}\eta = \frac{B^{(i)}_{m,2}\eta^3}{3} + \frac{B^{(i)}_{m,5}\eta^6}{6} + \frac{B^{(i)}_{m,8}\eta^9}{9} + \ldots \tag{I.3}$$

Study the behaviour of $q^{(i)}_m(\eta)$ at large η. It is known [3] that at large η

$$f(\eta) = \eta - b + 0\left(\frac{1}{\eta^2}\exp\{-\tfrac{1}{2}(\eta - b)^2\}\right),\ b = 1,2168\ldots, \tag{I.4}$$

It may be proved that equation (44) has the solutions which are of infinite continuation to the right. These solutions are stable and represented by the asymptotic series

$$q^{(i)}_m(\eta) = \eta + \sum_{n=0}^{+\infty} r^{(i)}_{m,n}\eta^{-n},\ (\eta \to \infty). \tag{I.5}$$

Having substituted (I.4) and (I.5) into (44) we have $r^{(i)}_{m,0} = -b$, $r^{(i)}_{m,1} = 2\mu^{(i)}_m/Pr$. Calculate the remainder of the coefficients we have the recurrent formulae

$$r^{(i)}_{m,n+1} = -\frac{n-1}{Pr}r^{(i)}_{m,n-1} - \frac{a^{(i)}_{m,n}}{2},\ n = 1,2,3,\ldots, \tag{I.6}$$

where

$$a^{(i)}_{m,n} = \begin{cases} \dfrac{1}{nr^{(i)}_{m,0}}\displaystyle\sum_{p=1}^{n}(3p - n)\,r^{(i)}_{m,p}a^{(i)}_{m,n-p} & \text{at } n \geq 1, \\ r^{(i)2}_{m,0} & \text{at } n = 0. \end{cases}$$

Note that for large η

$$Q^{(i)}_m(\eta) = \frac{\eta^2}{2} - b\eta + \frac{2\mu^{(i)}_m}{Pr}\ln\eta + \sum_{n=0}^{+\infty} l^{(i)}_{m,n}\eta^{-n}, \tag{I.7}$$

where $l^{(i)}_{m,n} = -(r^{(i)}_{m,n+1}/n)\,(n = 1,2,\ldots)$ and $l^{(i)}_{m,0}$ is determined by matching series (I.3) and (I.7).

Using the method of quasilinearization, the solution of (44) may be reduced to the solution of the following problem:

$$q^{(i)}_m(\eta) = \min_u\left\{\int_0^\eta \exp\left(2\int_{\eta'}^{\eta} u(x)\,\mathrm{d}x\right)\left[\frac{2}{Pr}u^2(\eta') - f^{(i)}_m(\eta')\right]\mathrm{d}\eta'\right.$$

where

$$\bar{f}_m^{(i)}(\eta) = (1 + 2\mu_m^{(i)})f'(\eta) + \frac{Pr}{2}f^2(\eta).$$

This problem may be solved by successive approximations on the computer. Here are the recurrent relations

$$\bar{q}_{m,n+1}^{(i)\prime} = \bar{q}_{m,n}^{(i)2} - 2\bar{q}_{m,n}^{(i)}\bar{q}_{m,n+1}^{(i)} + \bar{f}_m^{(i)}(\eta), \bar{q}_m^{(i)} = \frac{2}{Pr}q_m^{(i)}(\eta),$$

$$\bar{q}_{m,n+1}^{(i)}(0) = 0,$$

where $\bar{q}_{m,0}^{(i)}$ is the convenient initial approximation. These successive approximations possess two important properties which are rather powerful from the computational and analytical viewpoint: (1) quadratic convergence, (2) uniform monotonic convergence at any section $[0, a']$.

APPENDIX II

(a) *Approximate calculation of* $\psi_{j,k}^{(i)}(0)$, $\varkappa_{j,l}^{(i)}(0)$. Upon solving the system (53), (54) by the constant variation method we have

Substituting (II.6) and expansion $q_m^{(i)}(\eta)$, $Q_m^{(i)}(\eta)$ (see formulae (I.1), (I.3) from Appendix I) into (II.5) we find†

$$z_{j,m,2} = -(1-\delta_{0j})z_{j-1,m-1,0}; z_{j,m3} = -(1-\delta_{0j}) \times z_{j-1,m-1,1}; z_{j,m,4} = -Pr\lambda_m\alpha z_{j,m,1} - (1-\delta_{0j}) \times z_{j-1,m-1,2}; z_{j,m,5} = -3Pr\lambda_m\alpha z_{j,m,2} - (1-\delta_{0j}) \times (z_{j-1,m-1,3} - 2Pr\alpha z_{j-1,m-1,0}). \quad \text{(II.7)}$$

Now find the five terms in the expansion

$$z_{j-1,m-1}\, e^{(Pr/2)(Q_{m-1}+Q_m)} = \sum_{n=0}^{\infty} B_{j,m,n}\eta^n \quad \text{(II.8)}$$

$$\bar{B}_{j,m0} = z_{j-1,m-1,0}, \bar{B}_{j,m,1} = \frac{z_{j-1,m-1,1}}{1!}, \bar{B}_{j,m,2} = \frac{z_{j-1,m-1,2}}{2!}$$

$$\bar{B}_{j,m,3} = \frac{z_{j-1,m-1,3}}{3!} + z_{j-1,m-1,0}\frac{t_{m,3}\alpha}{3!},$$

$$\bar{B}_{j,m,4} = \frac{z_{j-1,m-1,4}}{4!} + \frac{z_{j-1,m-1,1}t_{m,3}\alpha}{3!},$$

$$\bar{B}_{j,m,5} = \frac{z_{j-1,m-1,5}}{5!} + \frac{z_{j-1,m-1,2}}{2!}\frac{t_{m,3}\alpha}{3!}, \quad \text{(II.9)}$$

$$\psi_{j,k}^{(i)}(0) = -\delta_{0j}\int_0^\infty e^{-PrF_k^{(i)}(\eta)}\,d\eta + (1-\delta_{0j})\int_0^\infty\left(\int_0^\eta \psi_{j-1,k-1}^{(i)}\, e^{(Pr/2)(F_{k-1}^{(i)}+F_k^{(i)})}d\eta'\right)e^{-PrF_k^{(i)}(\eta)}\,d\eta, \quad \text{(II.1)}$$

$$\varkappa_{j,l}^{(i)}(0) = \frac{(1-\delta_{0j})\int_0^\infty\left(\int_0^\eta \varkappa_{j-1,l-1}^{(i)}\, e^{(Pr/2)(P_{l-1}^{(i)}+P_l^{(i)})}d\eta'\right)e^{-PrP_l^{(i)}(\eta)}\,d\eta - \delta_{0j}}{\int_0^\infty e^{-PrP_l^{(i)}(\eta)}\,d\eta} \quad \text{(II.2)}$$

From (II.1) and (II.2) it is seen that the problem of calculating $\psi_{j,k}^{(i)}(0)$, $\varkappa_{j,l}^{(i)}(0)$ is reduced to calculating the integral form:

$$I_{j,m}^{(i)} = \int_0^\infty\left(\int_0^\eta z_{j-1,m-1}^{(i)}(\eta')\, e^{(Pr/2)(Q_{m-1}^{(i)}+Q_m^{(i)})}d\eta'\right)e^{-PrQ_m^{(i)}(\eta)}\,d\eta, \quad \text{(II.3)}$$

$$I_m^{(i)}\{Q\} = \int_0^\infty e^{-PrQ_m^{(i)}(\eta)}\,d\eta, \quad \text{(II.4)}$$

where the functions $Z_{j,m}^{(i)}(\eta)$ satisfy the system

$$\frac{d^2 z_{j,m}^{(i)}}{d\eta^2} + Prq_m^{(i)}(\eta)\frac{dz_{j,m}^{(i)}}{d\eta} = -(1-\delta_{0j})z_{j-1,m-1}^{(i)}(\eta)\, e^{(Pr/2)(Q_{m-1}^{(i)}(\eta)-Q_m^{(i)}(\eta))}. \quad \text{(II.5)}$$

We shall search for the first five terms expanding into series for $z_{j,m}^{(i)}(\eta)$

$$z_{jm}^{(i)}(\eta) = \sum_{n=0}^{\infty}\frac{z_{j,m,n}^{(i)}}{n!}\eta^n, z_{j,m,n}^{(i)} = \left.\frac{d^n z_{j,m}^{(i)}}{d\eta^n}\right|_{\eta=0}. \quad \text{(II.6)}$$

where

$$t_{m,3} = \tfrac{1}{2}Pr(\lambda_m + \lambda_{m-1}).$$

Hence it is easy to find the first terms of the expansion

$$\varphi_{j,m}(\eta) = \int_0^\eta z_{j-1,m-1}(\eta)\, e^{(Pr/2)(Q_{m-1}+Q_m)}\,d\eta = \sum_{n=1}^{\infty} b_{j,m,n}\eta^n. \quad \text{(II.10)}$$

It is obvious that

$$B_{j,m,n} = \frac{1}{n}\bar{B}_{j,m,n-1}. \quad \text{(II.11)}$$

Pass to the approximate calculation of the integral

$$I_{j,m} = \int_0^\infty \varphi_{j,m}(\eta)\, e^{-PrQ_m(\eta)}\,d\eta. \quad \text{(II.12)}$$

† Later on the subscript i is omitted (except the final results).

Let us use the ideas of one of the variants of the steepest descend method [3]. Let us make the replacement in (II.12)

$$\tau = Q_m(\eta) = \eta^3(C_{m,0} + C_{m,3}\eta^3 + C_{m,6}\eta^6), \tag{II.13}$$

$$C_{m,n} = \frac{1}{n+3} B_{m,n+2}. \tag{II.14}$$

For rather small τ the expansions are valid [3]

$$\eta = \sum_{r=0}^{\infty} \frac{A_{m,r}}{r+1} \tau^{\frac{1}{3}(r+1)}, \tag{II.15}$$

$$\varphi_{j,m} \frac{d\eta}{d\tau} = \tau^{-\frac{2}{3}} \sum_{r=0}^{\infty} d_{j,m,r} \tau^{r/3}, \tag{II.16}$$

$A_{m,r}$ is the coefficient at η^r in the expression

$$(C_{m,0} + C_{m,3}\eta^3 + C_{m,6}\eta^6 + \ldots)^{-\frac{1}{3}(r+1)}, \tag{II.17}$$

$d_{j,m,r}$ is equal to $\frac{1}{3}$ of the coefficient at η^r in the expression $(C_{m,0} + C_{m,3}\eta^3 + C_{m,6}\eta^6 + \ldots)^{-\frac{1}{3}(r+1)}$

$$(B_{j,m,1}\eta + B_{j,m,2}\eta^2 + B_{j,m,3}\eta^3 + \ldots) \tag{II.18}$$

Expanding expressions (II.17) and (II.18) into series gives:

$$A_{m,0} = C_{m,0}^{-\frac{1}{3}}, A_{m,1} = 0, A_{m,2} = 0, A_{m,3} = -\tfrac{4}{3}C_{m,0}^{-\frac{7}{3}}C_{m,3},$$
$$A_{m,4} = 0, A_{m,5} = 0,$$

$$A_{m,6} = C_{m,0}^{-\frac{7}{3}}\left[-\frac{7}{3}\frac{C_{m,6}}{C_{m,0}} + \frac{35}{9}\left(\frac{C_{m,3}}{C_{m,0}}\right)^2\right], \tag{II.19}$$

$$d_{j,m,0} = 0, d_{j,m,1} = \tfrac{1}{3}C_{m,0}^{-\frac{2}{3}}B_{j,m,1}, d_{j,m,2} = \tfrac{1}{3}C_{m,0}^{-1}B_{j,m,2},$$
$$d_{j,m,3} = \tfrac{1}{3}C_{m,0}^{-\frac{4}{3}}B_{j,m,3}, d_{j,m,4} = \tfrac{1}{3}C_{m,0}^{-\frac{5}{3}}$$
$$\times\left(B_{j,m,4} - \frac{5}{3}\frac{C_{m,3}}{C_{m,0}}B_{j,m,1}\right), d_{j,m,5} = \tfrac{1}{3}C_{m,0}^{-2}$$
$$\times\left({}_{j,m,5} - 2\frac{C_{m,3}}{C_{m,0}}{}_{j,m,2}\right), d_{j,m,6} = \tfrac{1}{3}C_{m,0}^{-\frac{7}{3}}$$
$$\times\left(B_{j,m,6} - \frac{7}{3}\frac{C_{m,3}}{C_{m,0}}B_{j,m,3}\right). \tag{II.20}$$

Thus, upon substitution of (II.16) into (II.12) with regard for (II.9), (II.11), (II.14) and (II.20), we have

$$I_{j,m} = \sum_{r=0}^{+\infty} d_{j,m,r}\Gamma\left(\frac{r+1}{3}\right)Pr^{-(r+1)/3}, d_{j,m,0} = 0,$$

$$d_{j,m,1} = \frac{z_{j-1,m-1,0}}{3}\left(\frac{6}{\lambda_m\alpha}\right)^{\frac{2}{3}}, d_{j,m,2} = \frac{z_{j-1,m-1,1}}{\lambda_m\alpha},$$

$$d_{j,m,3} = \frac{z_{j-1,m-1,2}}{3\lambda_m\alpha}\left(\frac{6}{\lambda_m\alpha}\right)^{\frac{1}{3}},$$

$$d_{j,m,4} = \left(\frac{6}{\lambda_m\alpha}\right)^{\frac{2}{3}}\frac{1}{12\lambda_m}\left\{\frac{z_{j-1,m-1,3}}{\alpha} + \left[\frac{1}{3} + \frac{Pr}{2}\right.\right.$$
$$\left.\left.\times\left(\frac{3\lambda_m^2 - 2}{\lambda_m} - \lambda_{m-1}\right)\right]z_{j-1,m-1,0}\right\}, d_{j,m,5} = \frac{1}{10\lambda_m^2\alpha}$$
$$\times\left\{\frac{z_{j-1,m-1,4}}{\alpha} + \left[1 + Pr\left(\frac{5\lambda_m^2 - 3}{\lambda_m} + 2\lambda_{m-1}\right)\right]\right.$$
$$\left.\times z_{j-1,m-1,1}\right\}, d_{j,m,6} = \left(\frac{6}{\lambda_m\alpha}\right)^{\frac{1}{3}}\frac{1}{60\lambda_m^2\alpha}\left\{\frac{z_{j-1,m-1,5}}{\alpha}\right.$$
$$\left. + \left[\frac{7}{3} + Pr\left(\frac{12\lambda_m^2 - 7}{\lambda_m} + 5\lambda_{m-1}\right)\right]z_{j-1,m-1,2}\right\}. \tag{II.21}$$

Similarly, substituting (II.15) into (II.4) with regard for (II.14) and (II.19), gives

$$I_m\{Q\} = \sum_{r=0}^{+\infty} d_{m,r}\Gamma\left(\frac{r+1}{3}\right)Pr^{-(r+1)/3}, \tag{II.22}$$

where

$$d_{m,0} = \frac{1}{3}\left(\frac{6}{\lambda_m\alpha}\right)^{\frac{1}{3}}, d_{m,1} = d_{m,2} = 0,$$

$$d_{m,3} = \frac{1}{3}\left(\frac{6}{\lambda_m\alpha}\right)^{\frac{1}{3}}\left[\frac{1}{15\lambda_m} + \frac{Pr(\lambda_m^2 - 1)}{5\lambda_m^2}\right], d_{m,4} = d_{m,5} = 0,$$

$$d_{m,6} = \frac{1}{60}\left(\frac{6}{\lambda_m\alpha}\right)^{\frac{1}{3}}\left\{-\frac{1}{9\lambda_m^2} + \frac{7Pr(\lambda_m^2 - 1)}{12\lambda_m^3} - \frac{7Pr^2(\lambda_m^2 - 1)}{4\lambda_m^4}\right\}.$$

Apply the general formulae obtained for calculating $\psi_{j,k}^{(i)}(0)$. Let

$$\psi_{j,k}^{(i)}(\eta) = \sum_{n=0}^{+\infty}\frac{r_{j,k,n}^{(i)}}{n!}\eta^n, \qquad r_{j,k,n}^{(i)} = \left.\frac{d^n\psi_{j,k}^{(i)}}{d\eta^n}\right|_{\eta=0}. \tag{II.23}$$

Substituting (II.21) into (II.1) and using (II.7) with regard for $\psi_{j,k}^{(i)'}(0) = \delta_{0,j}$ give

$$r_{j,k,0}^{(i)} = -\delta_{0,j}I_k^{(i)}\{F\} + (1 - \delta_{0,j})\sum_{r=1}^{+\infty} d_{j,k,r}^{(i)'}$$
$$\Gamma\left(\frac{r+1}{3}\right)Pr^{-(r+1)/3} \tag{II.24}$$

$$d_{j,k,1}^{(i)'} = \frac{r_{j-1,k-1,0}^{(i)}}{3}\left(\frac{6}{\lambda_k^{(i)}\alpha}\right)^{\frac{2}{3}}, \quad d_{j,k,2}^{(i)'} = \frac{\delta_{0,j-1}}{\lambda_k^{(i)}\alpha},$$

$$d_{j,k,3}^{(i)'} = -\frac{(1 - \delta_{0,j-1})\,r_{j-2,k-2,0}^{(i)}}{3\lambda_k^{(i)}\alpha}\left(\frac{6}{\lambda_k^{(i)}\alpha}\right)^{\frac{1}{3}},$$

$$d_{j,k,4}^{(i)'} = \left(\frac{6}{\lambda_k^{(i)}\alpha}\right)^{\frac{2}{3}}\frac{1}{12\lambda_k^{(i)}}\left\{-\frac{\delta_{0,j-2}}{\alpha}\right.$$
$$\left. + \left[\frac{1}{3} + \frac{Pr}{2}\left(\frac{3\lambda_k^{(i)} - 2}{\lambda_k^{(i)}} + \lambda_{k-1}^{(i)}\right)\right]r_{j-1,k-j,0}^{(i)}\right\},$$

$$d^{(i)'}_{j,k,5} = \frac{1}{10\lambda_k^{(i)2}\alpha}\left\{\frac{r^{(i)}_{j-1,k-1,4}}{\alpha} + \left[1 + Pr\left(\frac{5\lambda_k^{(i)2} - 3}{\lambda_k^{(i)}} + 2\lambda^{(i)}_{k-1}\right)\right]\delta_{0,j-1}\right\},$$

$$r^{(i)}_{j-1,k-1,4} = -\alpha\, Pr\, \lambda^{(i)}_{k-1}\,\delta_{0,j-1} + (1 - \delta_{0,j-1} - \delta_{0,j-2}) \times r^{(i)}_{j-3,k-3,0},$$

$$d^{(i)'}_{j,k,6} = \left(\frac{6}{\lambda_k^{(i)}\alpha}\right)^{\frac{1}{3}}\frac{1}{60\lambda_k^{(i)2}\alpha}\left\{\frac{r^{(i)}_{j-1,k-1,5}}{\alpha} - \left[\frac{7}{3} + Pr\left(\frac{12\lambda_k^{(i)2} - 7}{\lambda_k^{(i)}} + 5\lambda^{(i)}_{k-1}\right)\right] \times (1 - \delta_{0,j-1}) \times r^{(i)}_{j-2,k-2,0}\right\}.$$

$$r^{(i)}_{j-1,k-1,5} = \alpha\, Pr(1 - \delta_{0,j-1})(2 + 3\lambda^{(i)}_{k-1}) \times r^{(i)}_{j-2,k-2,0} + \delta_{0,j-3},$$

$$r^{(i)}_{j,k,1} = \delta_{0,j},\quad r^{(i)}_{j,k,2} = -(1 - \delta_{0,j})\, r^{(i)}_{j-1,k-1,0},\quad r^{(i)}_{j,k,3} = -\delta_{0,j-1},$$

$$r^{(i)}_{j,k,4} = -\alpha\, Pr\, \lambda_k^{(i)}\,\delta_{0,j} + (1 - \delta_{0,j} - \delta_{0,j-1})\, r^{(i)}_{j-2,k-2,0},$$

$$r^{(i)}_{j,k,5} = \alpha\, Pr(1 - \delta_{0,j})(2 + 3\lambda_k^{(i)})\, r^{(i)}_{j-1,k-1,0} + \delta_{0,j-2},$$

$$\lambda_k^{(i)} = 1 + 4(\rho^{(i)} + k + 1). \qquad \text{(II.24')}$$

Similarly apply the general formulae for calculating $\varkappa^{(i)'}_{jk}(0)$. Let

$$\varkappa^{(i)}_{j,k}(\eta) = \sum_{n=0}^{\infty}\frac{S^{(i)}_{j,k,n}}{n!}\eta^n,\quad S^{(i)}_{j,k,n} = \left.\frac{d^n\varkappa^{(i)}_{j,k}}{d\eta^n}\right|_{\eta=0}. \qquad \text{(II.25)}$$

Substituting (II.21) into (II.2) and using (II.7) with regard for $\varkappa^{(i)}_{j,n}(0) = \delta_{0,j}$ give

$$s^{(i)}_{j,l,1} = \frac{1}{I_l^{(i)}\{P\}}\left[-\delta_{0,j} + (1 - \delta_{0,j})\sum_{r=1}^{\infty} d^{(i)''}_{j,l,r}\,\Gamma\left(\frac{r+1}{3}\right) Pr^{-(r+1)/3}\right],$$

$$d^{(i)''}_{j,l,1} = \frac{\delta_{0,j-1}}{3}\left(\frac{6}{\lambda_l^{(i)}\alpha}\right)^{\frac{2}{3}},\quad d^{(i)''}_{j,l,2} = \frac{s^{(i)}_{j-1,l-1,1}}{\lambda_l^{(i)}\alpha},$$

$$d^{(i)''}_{j,l,3} = -\frac{\delta_{0,j-2}}{3\lambda_l^{(i)}\alpha}\left(\frac{6}{\lambda_l^{(i)}\alpha}\right)^{\frac{1}{3}},$$

$$d^{(i)''}_{j,l,4} = \left(\frac{6}{\lambda_l^{(i)}\alpha}\right)^{\frac{2}{3}}\frac{1}{12\lambda_l^{(i)}}\left\{\frac{-(1 - \delta_{0,j-1})\, s^{(i)}_{j-2,l-2,1}}{\alpha} + \delta_{0,j-1}\left[\frac{1}{3} + \frac{Pr}{2}\left(\frac{3\lambda_l^{(i)2} - 2}{\lambda_l^{(i)}} + \lambda^{(i)}_{l-1}\right)\right]\right\}, \qquad \text{(II.26')}$$

$$d^{(i)''}_{j,l,5} = \frac{1}{10\lambda_l^{(i)2}\alpha}\left\{\frac{s^{(i)}_{j-1,l-1,4}}{\alpha} + \left[1 + Pr\left(\frac{5\lambda_l^{(i)2} - 3}{\lambda_l^{(i)}} + 2\lambda^{(i)}_{l-1}\right)\right] s^{(i)}_{j-1,l-1,1}\right\},$$

$$s^{(i)}_{j-1,l-1,4} = -\alpha\, Pr\, \lambda^{(i)}_{l-1}\, s^{(i)}_{j-1,l-1,1} + \delta_{0,j-3},$$

$$d^{(i)''}_{j,l,6} = \left(\frac{6}{\lambda_l^{(i)}\alpha}\right)^{\frac{1}{3}}\frac{1}{60\lambda_l^{(i)2}\alpha}\left\{\frac{s^{(i)}_{j-1,l-1,5}}{\alpha} \left[\frac{7}{3} + Pr\left(\frac{12\lambda_l^{(i)} - 7}{\lambda_l^{(i)}} + 5\lambda^{(i)}_{l-1}\right)\right]\delta_{j-2}\right\},$$

$$s^{(i)}_{j-1,l-1,5} = (1 - \delta_{0,j-1} - \delta_{0,j-2})\, s^{(i)}_{j-3,l-3,1} + \delta_{0,j-2}\,\alpha\, Pr(2 + 3\lambda^{(i)}_{l-1}),$$

$$s^{(i)}_{j,l,0} = \delta_{0,j},\quad s^{(i)}_{j,l,2} = -\delta_{i,j-1},$$

$$s^{(i)}_{j,l,3} = -(1 - \delta_{0,j})\, s^{(i)}_{j-1,l-1,1},$$

$$s^{(i)}_{j,l,4} = -\alpha\, Pr\, \lambda_l^{(i)}\, s^{(i)}_{j,l,1} + \delta_{0,j-2},$$

$$s^{(i)}_{j,l,5} = (1 - \delta_{0,j} - \delta_{0,j-1})\, s^{(i)}_{j-2,l-2,1} + \delta_{0,j-1}\,\alpha\, Pr(2 + 3\lambda_l^{(i)}), \qquad \text{(II.26')}$$

$$\lambda_l^{(i)} = 1 + 4(\rho^{(i)} + l + \tfrac{1}{2}).$$

With the aid of the recurrent formulae (II.24) and (II.26) it is possible successively to calculate the desired values $r^{(i)}_{j,k,0} = \psi^{(i)}_{j,k}(0)$, $s^{(i)}_{j,l,1} = \varkappa^{(i)'}_{jl}(0)$ and from (II.24') and (II.26') to determine the first five terms in the expansions $\psi^{(i)}_{j,k}(\eta)$ and $\varkappa^{(i)}_{j,l}(\eta)$ for small η.

When calculating sums (II.24) and (II.26) approximately in the case of slow series convergence it is necessary to use the Euler transformation.

(b) *Asymptotic behaviour of $\psi^{(i)}_{j,k}(\eta)$ and $\varkappa^{(i)}_{j,l}(\eta)$ at large η.* The solution of system (II.5) is sought in the form

$$z^{(i)}_{j,m}(\eta) = u^{(i)}_{j,m}(\eta)\exp(-Pr\, Q_m^{(i)}(\eta)). \qquad \text{(II.27)}$$

Substituting (II.27) into (II.5) gives

$$u^{(i)''}_{j,m} - Pr\, q_m u^{(i)'}_{j,m} - Pr\, q_m^{(i)'} u^{(i)}_{jm} = -(1 - \delta_{0,j})\, u^{(i)'}_{j-1,m-1}\, e^{(Pr/2)(Q_m^{(i)} - Q^{(i)}_{m-1})}. \qquad \text{(II.28)}$$

Let us search for the main term in the asymptotic expansion $u^{(i)}_{j,m}$ at large η in the form

$$u^{(i)}_{j,m} \sim c^{(i)}_{j,m}\{z\}\,\eta^{\alpha_j}. \qquad \text{(II.29)}$$

Substituting (II.29) into (II.28) and taking into account (I.5), (I.7) (see Appendix I) give

$$c^{(i)}_{j,m}\{z\} = \frac{c^{(i)}_{j-1,m-1}\{z\}}{Pr(\alpha_j + 1)},\quad \alpha_j = \alpha_{j-1} + 2,$$

$$\gamma = 1,2,\ldots \text{ and } \alpha_0 = -1. \qquad \text{(II.30)}$$

With the help of (II.30) it is easy to show that

$$z^{(i)}_{j,m} \sim \frac{c^{(i)}_{0,m-j}\{z\}\,\eta^{2j-1}}{(2\,Pr)^j j!}\, e^{-Pr\,Q_m^{(i)}(\eta)}. \qquad \text{(II.31)}$$

Determine $c^{(i)}_{0,m}\{\psi\}$ and $c^{(i)}_{0,m}\{\varkappa\}$ for systems (53) and (54), respectively. From (53) and (54) it follows that

$$\psi_{0,m}^{(i)}(\eta) = -\int_{\eta}^{+\infty} e^{-Pr\,F_m^{(i)}(\eta)}\,d\eta,$$

$$\varkappa_{0,m}^{(i)}(\eta) = \frac{1}{I_m^{(i)}\{P\}}\int_{\eta}^{+\infty} e^{-Pr\,P_m^{(i)}(\eta)}\,d\eta.$$

Obviously the conditions

$$\lim_{\eta\to\infty}\frac{\dfrac{-\int_{\eta}^{+\infty} e^{-PrF_m^{(i)}(\eta)}\,d\eta}{\eta}}{\dfrac{c_{0,m}^{(i)}\{\psi\}\,e^{-Pr\,F_m^{(i)}(\eta)}}{\eta}} = 1,$$

$$\lim_{\eta\to\infty}\frac{\dfrac{\frac{1}{I_m^{(i)}\{P\}}\int_{\eta}^{+\infty} e^{-Pr\,P_m^{(i)}(\eta)}\,d\eta}{\eta}}{\dfrac{c_{0,m}^{(i)}\{\varkappa\}\,e^{-Pr\,P_m^{(i)}(\eta)}}{\eta}} = 1$$

should be satisfied. Eliminating these uncertainties according to the L'Hospital rule we find

$$c_{0,m}^{(i)}\{\psi\} = -\frac{1}{Pr}, \qquad c_{0,m}^{(i)}\{\varkappa\} = \frac{1}{I_m^{(i)}\{P\}\,Pr}.$$

Thus, we have

$$\psi_{j,m}^{(i)} \sim -\frac{\eta^{2j-1}\exp\{-Pr\,F_m^{(i)}(\eta)\}}{2^j\,Pr^{j+1}\,j!},$$

$$\varkappa_{j,m}^{(i)} \sim \frac{\eta^{2j-1}\exp\{-Pr\,P_m^{(i)}(\eta)\}}{I_{m-j}^{(i)}\{P\}\,2^j\,Pr^{j+1}\,j!}. \qquad \text{(II.32)}$$

APPENDIX III

The Expansion of $\Omega(\tau)$ into Series

In calculating we use that

$$\int_0^{\infty} e^{(-x^2/4)}\,D_{-\nu}(x)\,x^{\mu-1}\,dx = \frac{\pi^{\frac{1}{2}}\,\Gamma(\mu)}{2^{\frac{1}{2}(\mu+\nu)}\,\Gamma\left(\dfrac{\mu+\nu+1}{2}\right)},$$

$$Re\mu > 0 \qquad \text{(III.1)}$$

(see [4], Chapters 8 and 9).

Replacing $x = (2\tau)^{\frac{1}{2}}x'$ in (127) and substituting the series expansion $g_n(\tau)$ into (127) we arrive at

$$G_n(\tau) = \sum_{k=1}^{\sigma}\sum_{l=1}^{m_k}\sum_{s=0}^{\infty}\frac{c_{n,k,l}(r_k)\,C_{m_k-l+s}^{s}\,r_k^s}{(l-1)!\,\Gamma\left(\dfrac{m_k-l+s+1}{2}\right)}\,\tau^{(m_k+l-s+1)/2}. \qquad \text{(III.2)}$$

Replacing $x = (2\tau)^{\frac{1}{2}}x'$ in (128) and substituting formula (III.2) into (128) upon simple transformations give:

$$\Omega_n(\tau) = \tau^{(\rho+n)/2}\sum_{s=0}^{\infty}\Big(\sum_{k=1}^{\sigma}\lambda_{n,k,s}\Big)\,\tau^{s/4}, \qquad \text{(III.3)}$$

where

$$\lambda_{n,k,s} = \sum_{l=l_k}^{m_k}\frac{C_{n,k,l}(r_k)\,C_s^{m_k-l}\,r_k^{s+l-m_k}}{(l-1)!\,\Gamma[(s+2(\rho+n+2))/4]},$$

$$l^k = \begin{cases}1 & \text{at } s \geqslant m_k - 1,\\ m_k - s & \text{at } s < m_k - 1.\end{cases}$$

In the case of the simple polynomial roots $\Pi(u)$

$$\lambda_{n,k,s} = \frac{T_n(r_k)\,r_k^s}{\Pi_k(r_k)\Gamma[\{s+2(\rho+n+2)\}/4]}.$$

Using the d'Alambert sign, it is easy to show that series (III.3) converges at any τ.

Numerical calculations for the present work will be published in the *Journal of Engineering Physics* (U.S.S.R.)

TRANSFERT THERMIQUE CONJUGUE NON PERMANENT ENTRE UNE SURFACE SEMI-INFINIE ET UN ECOULEMENT INCIDENT DE FLUIDE COMPRESSIBLE—I. REDUCTION A LA RELATION INTEGRALE

Résumé— La solution analytique du problème conjugué du transfert thermique non permanent dans une plaque semi-infinie avec sources sous la forme de séries entières généralisées est obtenue dans l'espace des fonctions généralisées (au sens de Sobolev–Schwartz).

On a trouvé le paramètre adimensionnel $B = (1/Re)(a_s/Ca_\infty)$ ayant un sens physique qui permet de considérer le problème comme non permanent ou quasi-stationnaire dépendant du couple "obstacle-liquide".

INSTATIONÄRE KONJUGIERTE WÄRMEÜBERTRAGUNG ZWISCHEN EINER HALBUNENDLICHEN OBERFLÄCHE UND EINEM ANSTRÖMENDEN KOMPRESSIBLEN FLUID
II. BESTIMMUNG EINES TEMPERATURFELDS UND ANALYSE VON ERGEBNISSEN

Zusammenfassung—Die analytische Lösung des konjugierten instationären Wärmeübertragungsproblems wurde in Form verallgemeinerter Potenzreihen aus dem Raum verallgemeinerter Funktionen (im Sobolew–Schwartz'schen Sinne) erhalten. Es wurde ein dimensionsloser Parameter $B = [1/Re . a_s/Ca]$ mit physikalischer Bedeutung gefunden, der eine Betrachtung des instationären und quasistationären Problems ermöglicht und nur von der Paarung "Körper–Flüssigkeit" abhängt.

НЕСТАЦИОНАРНЫЙ СОПРЯЖЕННЫЙ ТЕПЛООБМЕН ПОЛУБЕСКОНЕЧНОЙ ПОВЕРХНОСТИ С НАТЕКАЮЩИМ ПОТОКОМ СЖИМАЕМОЙ ЖИДКОСТИ II. ОПРЕДЕЛЕНИЕ ТЕМПЕРАТУРНОГО ПОЛЯ И АНАЛИЗ РЕЗУЛЬТАТОВ

Аннотация—В пространстве обобщенных функций в смысле Соболева-Шварца получено аналитическое решение задачи сопряженного нестационарного теплообмена полубесконечной пластины с источниками в виде обобщенного степеного ряда.

Выделен обладающий физическим смыслом безразмерный параметр

$$B = \left[\frac{1}{\mathrm{R}e} \cdot \frac{a_s}{Ca_\infty}\right],$$

который позволяет рассматривать задачу как нестационарную или квазистационарную в зависимости от пары «тело–жидкость».